ATLAS Note

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Derivation of the luminosity algorithms

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1 Introduction

The purpose of this note is to explain in a detailed step-by-step way how the different algorithms used in the luminosity analysis were derived. It also discusses possible systematic errors that are introduced in the analysis when these algorithms are used.

2 The Event OR Algorithm

2.1 Derivation of the algorithm

In order to obtain the probability functions for different classes of events, one can start with defining four inclusive efficiencies (or probabilities) for detecting different types of events when there is exactly one interaction per bunch crossing:

ε_A	The efficiency for detecting interactions with at least one hit on side A
ε_C	The efficiency for detecting interactions with at least one hit on side C
ε_{AND}	The efficiency for detecting interactions with at least one hit on side A and C
ε_{OR}	The efficiency for detecting interactions with at least one hit on side A or C

(1)

With these efficiencies one has $\varepsilon_A + \varepsilon_C = \varepsilon_{AND} + \varepsilon_{OR}$. Another way of describing the single interaction events is by a set of four exclusive probabilitites:

p_{10}	$= \varepsilon_A - \varepsilon_{AND}$	The probability of detecting an interaction in A, but not in C
p_{01}	$= \varepsilon_C - \varepsilon_{AND}$	The probability of detecting an interaction in C, but not in A
p_{11}	$= \varepsilon_{AND}$	The probability of detecting an interaction in both modules
p_{00}	$= 1 - \varepsilon_{OR}$	The probability of not detecting an interaction in either A or C

(2)

The relationship between these probabilities is: $p_{10} + p_{01} + p_{11} + p_{00} = 1$. After defining efficiencies/probabilities for a single interaction it is easy to define the detection probabilities for multi-interaction events. Under the assumption that the single interaction probabilities do not change when there are several interactions taking place in a short time period, one obtains the following probability for not detecting an event if there is exactly n interactions:

$$P_{00}(n) \quad = p_{00}^n = (1 - \varepsilon_{OR})^n \tag{3}$$

Now assume that n is a Poissonian distributed quantity with an average value called μ. The probability to get no hits in both detectors as a function of μ is then:

$$P_{00}(\mu) = \sum_{n=0}^{\infty} (1 - \varepsilon_{OR})^n \frac{e^{-\mu}\mu^n}{n!} = e^{-\varepsilon_{OR}\mu} \tag{4}$$

since the Maclaurin series expansion is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{5}$$

The probability to have a single side trigger with at least one recorded hit in a bunch crossing with on average μ interactions is thus

$$P_{OR}(\mu) = 1 - P_{00}(\mu) = 1 - e^{-\varepsilon_{OR}\mu} \tag{6}$$

By defining two new variable $\mu^{vis} \equiv \varepsilon\mu$ and $\sigma_{vis} \equiv \varepsilon\sigma_{inel}$ it is trivial to see that the bunch crossing luminosity ($\mathscr{L}_{BC}$) can be obtained from

$$\mathscr{L}_{BC} = \frac{\mu}{\sigma_{inel}} = \frac{\mu_{vis}}{\sigma_{vis}} \tag{7}$$

and that one can get the following relationship between the measured number of OR events (N_{OR}) and μ:

$$\tfrac{N_{OR}}{N_{BC}} = 1 - e^{-\varepsilon_{OR}\mu} = 1 - e^{-\mu_{vis}} \tag{8}$$

where N_{BC} is the number of bunch crossings that occurred during the measurement of N_{OR}. This expression is plotted in Figure 1 and compared to a simulation in which Monte Carlo data has been piled up to get different μ values.

Solving for μ_{vis} in terms of the event-counting rate yields:

$$\mu_{vis} = -\ln\left(1 - \tfrac{N_{OR}}{N_{BC}}\right) \tag{9}$$

and the luminosity for one bunch crossing is now given by

$$\mathscr{L}_{BC} = \frac{-\ln\left(1 - \frac{N_{OR}}{N_{BC}}\right)}{\sigma_{vis}} \tag{10}$$

A Taylor expansion of the logarithms gives

$$\ln\left(1 - \frac{N_{OR}}{N_{BC}}\right) = -\sum_{n=1}^{\infty} \frac{\left(\frac{N_{OR}}{N_{BC}}\right)^n}{n} \tag{11}$$

and so to first order in the expansion one gets

$$\mu_{vis} = \frac{N_{OR}}{N_{BC}} \tag{12}$$

With other words, a linear relationship between the luminosity and the number of events per BC is obtained if $N_{OR/BC} = \frac{N_{OR}}{N_{BC}}$ is much smaller than one.

2.2 Calibration of the Event OR Algorithm

In the VDM scans the luminosity at the peak of the scan distributions ($\mathscr{L}^{peak}$) is obtained from the widths of these distributions and the beam currents. The rate of events at the peak ($N^{peak}_{OR/BC}$) is also measured and from these two values it is possible to obtain a calibration constant, σ_{OR}, which is the visible inelastic cross section ($\sigma_{vis} \equiv \varepsilon\sigma_{inel} = \sigma_{OR}$) when the OR method is being used:

$$\sigma_{OR} = \frac{-\ln\left(1 - \frac{N_{OR}}{N_{BC}}\right)}{\mathscr{L}^{peak}} \tag{13}$$

If $N^{peak}_{OR/BC}$ (or μ^{peak}) is $<< 1$ during the VDM scan, then a linear approximation can also be used to obtain the visible cross section:

$$\sigma_{OR} = \tfrac{N_{OR/BC}}{\mathscr{L}^{peak}} \tag{14}$$

The final expression for the luminosity after calibration is thus:

$$\mathscr{L}_{BC} = \frac{-\ln\left(1 - \frac{N_{OR}}{N_{BC}}\right)}{\sigma_{OR}} \tag{15}$$

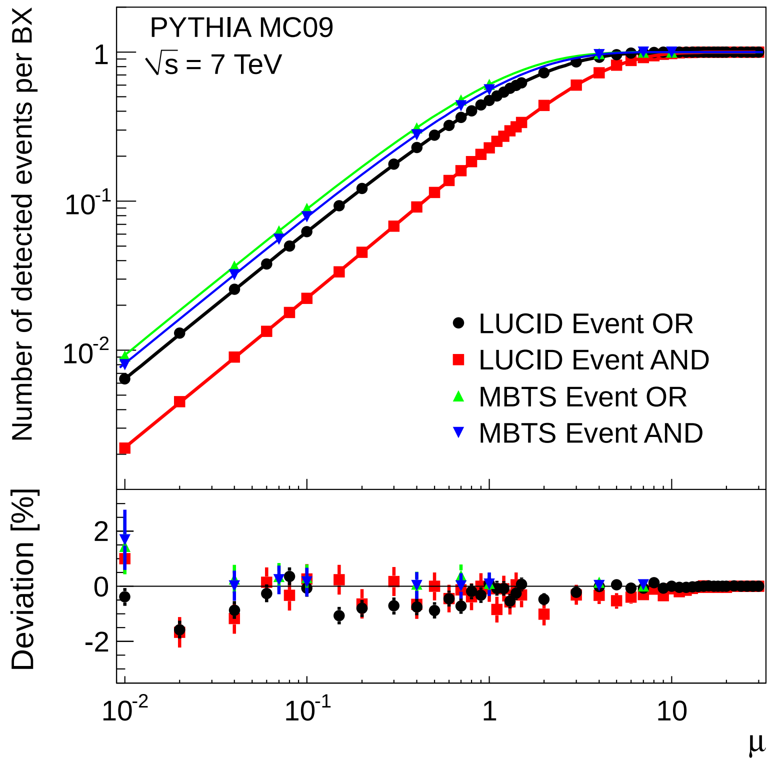

Figure 1: Comparison of probability functions with simulated data. The lines are from the equations and the points from the data. The LUCID OR-counting method is using Equation 8, while for LUCID AND-counting Equation 25 is used.

3 The Event AND Algorithm

3.1 Derivation of the algorithm

The same strategy can be used to obtain the probability to have a coincidence trigger. The first step is to note that the probability to have a coincidence event in exactly one interaction is

$$p_{11} = 1 - (p_{00} + p_{10} + p_{01}) \tag{16}$$

The second step is to calculate what the probability is for exactly n interactions, $P_{11}(n)$. Under the assumption that the probabilities to observe a single interaction is the same also in multi-interaction events one has:

$$P_{11}(n) = 1 - (P_{00}(n) + P_{10}(n) + P_{01}(n)) \tag{17}$$

$P_{00}(n)$ is trivial since:

$$P_{00}(n) = p_{00}^{n} \tag{18}$$

The terms $P_{10}(n)$ and $P_{01}(n)$ are a bit more complicated since one has to take into account all permutations of k interactions detected in module A (C) and $n-k$ interactions not detected in any module:

$$P_{10}(n) = \sum_{k=1}^{n} p_{10}^{k}\; p_{00}^{n-k} \binom{n}{k} = (p_{10}+p_{00})^n - p_{00}^n \tag{19}$$

$$P_{01}(n) = \sum_{k=1}^{n} p_{01}^{k}\; p_{00}^{n-k} \binom{n}{k} = (p_{01}+p_{00})^n - p_{00}^n \tag{20}$$

These expressions can now be used to obtain $P_{11}(n)$:

$$P_{11}(n) = 1-(P_{00}(n)+P_{10}(n)+P_{01}(n)) = 1-(p_{00}^n+(p_{10}+p_{00})^n - p_{00}^n+(p_{01}+p_{00})^n - p_{00}^n) \tag{21}$$

The final third step is to compute the probability if there are on average μ interactions in a bunch crossing by assuming a Poisson distribution:

$$P_{11}(\mu) = 1-(\sum_{n=0}^{\infty}(p_{10}+p_{00})^n \frac{e^{-\mu}\mu^n}{n!} + \sum_{n=0}^{\infty}(p_{01}+p_{00})^n \frac{e^{-\mu}\mu^n}{n!} - \sum_{n=0}^{\infty} p_{00}^n \frac{e^{-\mu}\mu^n}{n!}) \tag{22}$$

$$P_{11}(\mu) = e^{-\mu(1-p_{10}-p_{00})} + e^{-\mu(1-p_{01}-p_{00})} - e^{-\mu(1-p_{00})} \tag{23}$$

After substituting the probabilities for efficiencies one gets:

$$P_{AND}(\mu) = e^{-\mu\varepsilon_A} + e^{-\mu\varepsilon_C} - e^{-\mu(\varepsilon_A+\varepsilon_C-\varepsilon_{AND})} \tag{24}$$

Also in this case one can estimate the probability from the measured number of events (N_{AND}):

$$\frac{N_{AND}}{N_{BC}} = 1-(e^{-\mu\varepsilon_A} + e^{-\mu\varepsilon_C} - e^{-\mu(\varepsilon_A+\varepsilon_C-\varepsilon_{AND})}) \tag{25}$$

This expression is plotted in Figure 1 and compared to a simulation.

A Maclaurin expansion of the exponentials to first order gives:

$$\frac{N_{AND}}{N_{BC}} = \mu\varepsilon_{AND} = \mu_{vis} \tag{26}$$

and so for small μ values one again gets a linear relationship between the number of events and μ.

Equation 25 can be simplified in another way if one assumes that $\varepsilon_A = \varepsilon_C$ since in this case $\varepsilon_A = \varepsilon_C = (\varepsilon_{OR}+\varepsilon_{AND})/2$ and $\varepsilon_A+\varepsilon_C-\varepsilon_{AND} = \varepsilon_{OR}$:

$$\frac{N_{AND}}{N_{BC}} = 1-2e^{-\mu(\varepsilon_{OR}+\varepsilon_{AND})/2} + e^{-\mu\varepsilon_{OR}} \tag{27}$$

By again introducing $\mu_{vis} \equiv \mu\varepsilon_{AND}$ and the two visible cross sections $\sigma_{OR} \equiv \varepsilon_{OR}\sigma_{inel}$ and $\sigma_{AND} \equiv \varepsilon_{AND}\sigma_{inel}$ one gets

$$\frac{N_{AND}}{N_{BC}} = 1-2e^{-(\frac{\sigma_{OR}}{\sigma_{AND}}+1)\mu_{vis}/2} + e^{-\frac{\sigma_{OR}}{\sigma_{AND}}\mu_{vis}} \tag{28}$$

Even this simplified equation cannot be inverted analytically. The inverted distribution can, however, be turned into a look-up table or fitted with a third order polynomial:

$$\mu_{vis} \approx \alpha_1 \tfrac{N_{AND}}{N_{BC}} + \alpha_2(\tfrac{N_{AND}}{N_{BC}})^2 + \alpha_3(\tfrac{N_{AND}}{N_{BC}})^3 \tag{29}$$

where α_1, α_2 and α_3 are fitting constants. The luminosity is as previously given by:

$$\mathscr{L}_{BC} = \frac{\mu_{vis}}{\sigma_{AND}} \tag{30}$$

3.2 Calibration of the Event AND Algorithm

The VDM calibration for the AND case is more difficult than the OR case since there are two calibration constants in Equation 28 and the equation cannot be inverted. An iterative procedure has to be used where μ_{vis} at the peak of the VDM scan is calculated from Equations 28 and 29 using σ_{AND} from a Monte Carlo simulation or from a previous VDM scan. A new σ_{AND} value can then be calculated from

$$\sigma_{AND} = \frac{\mu_{vis}^{peak}}{\mathscr{L}^{peak}} \tag{31}$$

A second iteration using this new value of σ_{AND} can be put into Equation 28 which can be inverted to Equation 29 which then can be used to calculate a new value of σ_{AND} using again Equation 31.

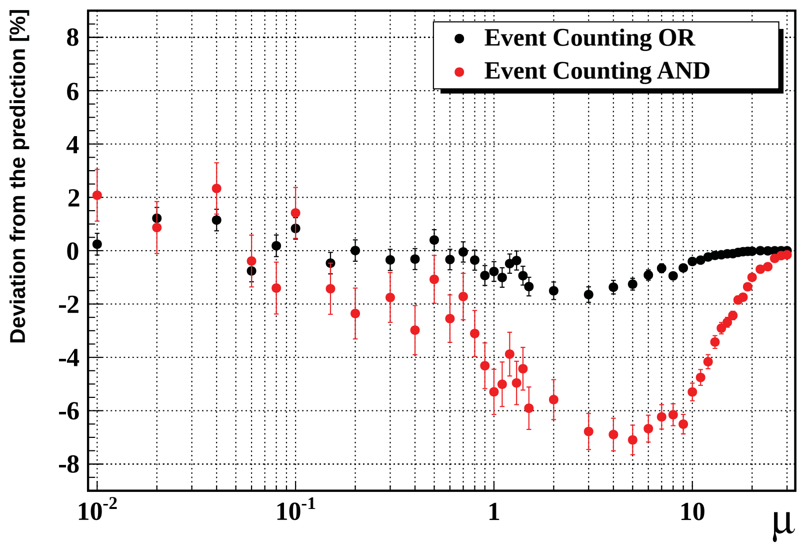

Figure 2: The difference bewteen the simulated data and Equation 8 and 25 when the LUCID threshold is increased to 50 photoelectrons.

4 Systematic uncertainties in the event counting algorithms

The shapes of Equations 8 and 25 are shown in Figure 1. As μ increases, the number of events per bunch crossing will eventually become close to one. When this happens the methods become unreliable because a small change in counting rate then translates into a large change in luminosity. This behavior of the event counting methods is called "saturation" because the event rate cannot increase further when there is at least one detected interaction for every bunch crossing. A method with a high efficiency to detect a single interaction will saturate at a lower μ than a method with a higher efficiency. The saturation effect is one reason why hit-counting methods have been considered as alternative methods to event counting since they could in principle extend the range of the μ-measurement without saturation to higher values.

A comparison has also been made of the probability functions discussed above with simulated events that were generated by Pythia MC09 and then passed through the full ATLAS detector simulation. These events have been piled up according to a Poissonian distribution to give samples with different average

μ-vaules. If this pile-up results in a multi-interaction event, all the signals from the particles of the individual pp-interactions are added up channel-by-channel to produce a realistic pile-up event. The OR and AND selection is then made on these simulated pile-up events and the event rate is estimated. The difference between these simulated events and the Equations 8 and 25 is less than 2% for all cases and are typically less than 1%.

The good agreement between the simulated data and the probability functions means that it is correct to assume, as it was done above, that the detection efficiency for an individual pp-interaction is the same if there is a single or several interactions in a bunch crossing. That this is not an assumption that is always true can be seen in Figure 2. It shows the same difference between the probability function and the simulated LUCID data as in the previous figure but now with an increased discriminator threshold. A particle going through the center of a LUCID tube typically gives a signal that corresponds to 100 photo electrons and the discriminator threshold that has been used corresponds to 15 photo electrons. This threshold has been increased in the simulation in Figure 2 to 50 photo electrons and then there are deviations from the probability functions of up to 7% . The reason for this is that there are many secondary particles that do not go through the center of the LUCID tubes and which therefore produce a continuous spectrum of signals with a pulse height from zero up to several hundred photo electrons. The particles that give a signal below the discriminator threshold in one pp-interaction are able to pile-up and give a signal above threshold in multi-interaction events. This effect is called migration since particles with low pulse height are said to ”migrate” to a larger pulseheight in multi-interaction events.

Another source of systematic errors is caused by the numerical inversion of the probability functions. When a third-degree polynomial fit is used as in Equation 29 this typically introduces a maximum error of around 1% as long as the μ-range is limited to less than 5. Two fits in different μ-ranges or look-up tables can be used to obtain a higher precision.

5 The HIT OR Algorithms

5.1 Derivation of the algorithm

The average number of pp collisions per event (μ) can in OR mode also be calculated from the ratio between the average number of detected particles per bunch crossing ($N_{part/BC}$) and those detected in one pp collision ($N_{part/pp}$):

$$\mu = \frac{N_{part/BC}}{N_{part/pp}} \tag{32}$$

The average bunch luminosity is in this case given by

$$\mathscr{L}_{BC} = \frac{\mu}{\sigma_{inel}} = \frac{N_{part/BC}}{\sigma_{inel} N_{part/pp}} \tag{33}$$

After a calibration that has yielded $\sigma_{inel} N_{part/pp}$ it is thus possible to calculate the luminosity from a measurement of the average number of detected particles per bunch crossing.

The LUCID detector counts hits and not particles and at high μ there is a significant probability that several particles go through one tube but produces only one hit. Equation 32 has to be corrected for this saturation effect and that can be done under two assumptions:

1. The particles are distributed with an equal probability over all tubes;
2. The number of particles that go through a tube follows a Poissonian distribution.

If $N_{part/pp}$ is the total number of detected particles in one pp-interaction and N_{tubes} is the total number of tubes in the detector (30) then the average number of detected particles per tube is simply $N_{part/pp}/N_{tubes}$. The assumption that the particles in a tube are distributed according to a Poissonian, means that the number of hits can be written as the product of the number of tubes times the probability to have at least one detected particle in a tube (namely a hit):

$$N_{hits/pp} = N_{tubes}\left[1 - e^{-\frac{N_{part/pp}}{N_{tubes}}}\right] \tag{34}$$

Equation 34 can be inverted to turn the number of hits into particles:

$$N_{part/pp} = -N_{tubes}\ln\left(1 - \frac{N_{hits/pp}}{N_{tubes}}\right) \tag{35}$$

Note that Equation 35 holds also for bunch crossings with several interactions:

$$N_{part/BC} = -N_{tubes}\ln\left(1 - \frac{N_{hits/BC}}{N_{tubes}}\right) \tag{36}$$

and so one can obtain μ from a measurement of $N_{hits/BC}$ by using the following relationship:

$$\mu = \frac{N_{part/BC}}{N_{part/pp}} = \frac{\ln\left(1 - \frac{N_{hits/BC}}{N_{tubes}}\right)}{\ln\left(1 - \frac{N_{hits/pp}}{N_{tubes}}\right)} \tag{37}$$

One problem with this expression is that if the number of hits per BC is equal to the number of tubes, the luminosity becomes infinitely large. It is therefore necessary to take care of these cases in some way.

A Taylor expansion of the logarithms gives

$$\ln\left(1 - \frac{N_{hits}}{N_{tubes}}\right) = -\sum_{n=1}^{\infty}\frac{\left(\frac{N_{hits}}{N_{tubes}}\right)^n}{n} \tag{38}$$

and so to first order in the expansion one gets

$$\mu = \frac{N_{part/BC}}{N_{part/pp}} = \frac{N_{hits/BC}}{N_{hits/pp}} \tag{39}$$

With other words, if the ratio of the number of hits to the number of tubes is small then the logarithmic formula which takes care of the problem of saturation is not needed.

The luminosity for one bunch crossing, including saturation effects, is then calculated from Equation 37:

$$\mathscr{L}_{BC} = \frac{\mu}{\sigma_{inel}} = \frac{\ln\left(1 - \frac{N_{hits/Bc}}{N_{tubes}}\right)}{\sigma_{inel}\ln\left(1 - \frac{N_{hits/pp}}{N_{tubes}}\right)} \tag{40}$$

5.2 Calibration of the Hit OR Algorithms

In the VDM scans the luminosity at the peak of the scan distributions ($\mathscr{L}^{peak}$) is obtained from the widths of these distributions and the beam currents. The rate at the peak ($N^{peak}_{hits/BC}$) is also measured and from these two values it is possible to obtain a calibration constant that has barn as unit and which therefore is called an effective visible cross section (σ^{hits}_{vis}).

$$\sigma_{vis}^{hits} = -\sigma_{inel} \ln\left(1 - \frac{N_{hits/pp}}{N_{tubes}}\right) = \frac{\ln\left(1 - \frac{N_{hits/BC}^{peak}}{N_{tubes}}\right)}{\mathscr{L}^{peak}} \tag{41}$$

The results of the VDM scan during the spring is $\sigma_{vis}^{hits} = 3.80 \pm 0.02 mb$.

This visible cross section can be inserted into Equation 40 and the average luminosity per BC can now be calculated using:

$$\mathscr{L}_{BC} = \frac{\mu}{\sigma_{inel}} = \frac{-\ln\left(1 - \frac{N_{hits/BC}}{N_{tubes}}\right)}{\sigma_{vis}^{hits}} \tag{42}$$

Another way of doing the calibration is to use the linear approximation as a starting point:

$$\mathscr{L}_{BC} = \frac{N_{hits/BC}}{\sigma_{inel} N_{hits/pp}} \tag{43}$$

A calibration constant called K_{cal} is again measured from the rate and luminosity at the peak of the scan distributions:

$$K_{cal} = \sigma_{inel} N_{hits/pp} = \frac{N_{hits/BC}^{peak}}{\mathscr{L}^{peak}} \tag{44}$$

The results of the VDM scan during the spring is $K_{cal} = 113.3 mb$.

This calibration constant can then be inserted into Equation 40 and one obtains

$$\mathscr{L}_{BC} = \frac{\ln\left(1 - \frac{N_{hits/BC}}{N_{tubes}}\right)}{\sigma_{inel} \ln\left(1 - \frac{K_{cal}}{\sigma_{inel} N_{tubes}}\right)} \tag{45}$$

The denominator contains not only the calibration constant but also the inelastic cross section which is not determined in the VDM scans. So a disadvantage with this method is that a value for the inelastic cross section (from Monte Carlo calculations) have to be used when the luminosity is calculated.

However, another Taylor expansion gives:

$$\sigma_{inel} \ln\left(1 - \frac{K_{cal}}{\sigma_{inel} N_{tubes}}\right) = -\sigma_{inel} \sum_{n=1}^{\infty} \frac{\left(\frac{K_{cal}}{\sigma_{inel} N_{tubes}}\right)^n}{n} \tag{46}$$

which means that to first order the inelastic cross section is cancelled out in the calculation:

$$\sigma_{inel} \ln\left(1 - \frac{K_{cal}}{\sigma_{inel} N_{tubes}}\right) = -\frac{K_{cal}}{N_{tubes}} \tag{47}$$

That the dependence on the simulation is small can also be seen if one calculate σ_{vis}^{hits} using the inelastic cross section from Pythia (71.5 mb) and Phojet (76.2 mb). The result is $\sigma_{vis}^{hits} = 3.88(3.87)$ for Pythia (Phojet). These numbers are, however, both 2% higher than the value of 3.80 that was obtained in the VDM scan.

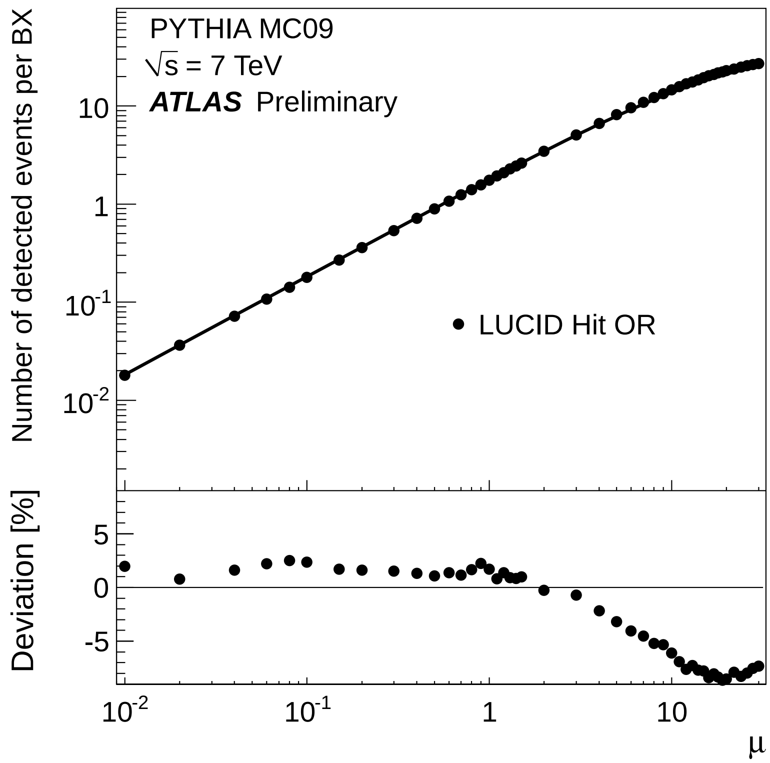

Figure 3: Comparison of hit functions with simulated data. The line is from Equation 37 after it has been inverted and the points are from simulated data.

6 Systematic uncertainties in the Hit OR counting algorithm

The line in Figure 3 shows Equation 37 after it has been inverted, i.e., it shows

$$N_{hits/BC} = N_{tubes}\left(1 - e^{-\mu K_{calib}}\right) \tag{48}$$

where K_{calib} is a calibration constant give by:

$$K_{calib} = \ln\left(1 - \frac{N_{hits/pp}}{N_{tubes}}\right) \tag{49}$$

The main reason for the large difference between Equation 48 and the Monte Carlo data in Figure 3 is most likely due to the assumption that the number of particles in LUCID are following a Poisson distribution. This assumption is fundamental when Equation 34 is used to derive the logarithmic formula that is used to measure the luminosity with the LUCID Hit-OR method. We know that the hit distributions in both real and simulated data are not Poisson distributed. This does not prove that the particle distributions are not following a Poisson but it is a clear indication that this is the case. So far we have not verified the Poisson assumption with simulated data. Another argument against this assumption is that LUCID detects a mixture of single and double diffractive events as well as inelastic events. We know that these different processes creates different hit distributions with different average number of hits. A sum of three Poisson distributions will not necessarily result in another distribution that is Poissonian.

The hit method will in addition suffer from the same migration effects that were discussed above for the event counting methods. How much of the discrepancy in Figure 3 is due to migration and how much is due to the Poissonian assumption is at present not known.

Another complication is due to the fact that Equation 34 is true only for one bunch crossing. One should therefore in principle use Equation 40 first on individual bunch crossings and then do the average and not use a value of $N_{hits/BC}$ that is obtained by averaging over all bunch crossing during a luminosity block. One should with other words use the following expression when calculating the luminosity:

$$\mathscr{L}_{BC} = \frac{1}{N_{Bc}} \cdot \frac{\sum_{i=1}^{N_{BC}} \ln\left(1 - \frac{N_{hits}(i)}{N_{tubes}}\right)}{\sigma_{inel} \ln\left(1 - \frac{N_{hits/pp}}{N_{tubes}}\right)} \quad (50)$$

This has, however, not been done in ATLAS because all rates are integrated over a luminosity block and then transformed to averaged rates that are put into Equation 37. This problem could in principle be corrected for in the LUMAT card but it has not been done.

In conclusion one can say that the formula used in the Hit-Or case is derived using very questionable assumptions and one should not be surprised that the data corrected with Equation 40 do not give a precise determination of the luminosity.

7 The HIT AND Algorithms

7.1 Derivation of the algorithm

In coincidence mode, there are two possibilities to detect an event with multiple interactions. A *true* coincidence occurs when at least one interaction is detected simultaneously in both modules. A *fake* coincidence occurs when no interaction is detected simultaneously in both modules, but at least two interactions are separately detected in different modules.

In coincidence mode, the average number of detected particles in events with n interactions is the sum of two contributions:

1. the event contains at least one interaction which is detected in both modules, together with any number of interactions which are only detected in module A and not in C, and vice versa;

2. the event contains 0 interactions detected in both modules, together with at least one interaction which is only detected in module A and one which is only detected in module C.

The average number of particles corresponding to Terms I and II is the sum of the probability of each configuration times the corresponding number of detected interactions, times the number of particles per detected interaction.

Definitions Four exclusive definitions of average number of particles in the whole detector per detected interaction are used:

$N^{01}_{part/pp*}$	no. of particles per detected interaction in A if there are no hits in C
$N^{10}_{part/pp*}$	no. of particles per detected interaction in C if there are no hits in A
$N^{11}_{part/pp*}$	no. of particles per detected interaction if there are hits in both modules
$N^{XOR}_{part/pp*}$	no. of particles per detected interaction if there are hits in one module but not both

(51)

Interactions that did not produce any particles in LUCID are not used in these averages. This is symbolised with a *. These averages will by definition be large than one (or two in the case of a coincidence requirement).

The probability of each configuration is evaluated by using the exclusive probabilities to detect an interaction defined above (p_{01}, p_{10}, p_{11} and p_{00}), together with the exlusive probability to detect an interaction in one module, but not in both ($p_{XOR} = p_{01} + p_{10}$). One can define a second set of average number of particles that are averaged also over those interactions that do not have particels in LUCID:

$N^{01}_{part/pp}$	$= p_{01} N^{01}_{part/pp*}$	no. of particles in A if there are no hits in C
$N^{10}_{part/pp}$	$= p_{10} N^{10}_{part/pp*}$	no. of particles in C if there are no hits in A
$N^{11}_{part/pp}$	$= p_{11} N^{11}_{part/pp*}$	no. of particles if there are hits in both modules
$N^{XOR}_{part/pp}$	$= p_{XOR} N^{XOR}_{part/pp*}$	no. of particles if there are hits in one module but not both

(52)

Given that $p_{01} N^{01}_{part/pp*}$ is the number of particles registered in the whole detector when the interaction is detected in module A only and $p_{10} N^{10}_{part/pp*}$ is the number of particles registered in the whole detector when the interaction is detected in module C only, the sum of these Terms gives the number of particles registered in the whole detector when the interaction is detected in module A or in module C but not in both ($p_{XOR} N^{XOR}_{part/pp*}$):

$$p_{XOR} N^{XOR}_{part/pp*} = p_{01} N^{01}_{part/pp*} + p_{10} N^{10}_{part/pp*} \tag{53}$$

One can also introduce a set of inclusive average number of particles if there are particles in LUCID:

N^{A}_{part/pp^*}	no. of particles per interaction detected in A (regardless of C)
N^{C}_{part/pp^*}	no. of particles per interaction detected in C (regardless of A)
N^{AND}_{part/pp^*}	no. of particles per interaction detected in both modules

(54)

And another inclusive set where one has averaged also over interactions with no particles in LUCID:

$N^{A}_{part/pp}$	$= \varepsilon_A N^{A}_{part/pp^*}$	no. of particles per interaction in A (regardless of C)
$N^{C}_{part/pp}$	$= \varepsilon_C N^{C}_{part/pp^*}$	no. of particles per interaction in C (regardless of A)
$N^{AND}_{part/pp}$	$= \varepsilon_{AND} N^{AND}_{part/pp^*}$	no. of particles per interaction in both modules

(55)

Term I Suppose n interactions occurred in an event, Term I can be written as:

$$I = \sum_{k=1}^{n} p_{11}^{k} \binom{n}{k} \left[\sum_{l=0}^{n-k} p_{XOR}^{l} (1 - p_{XOR} - p_{11})^{n-k-l} \binom{n-k}{l} \right] [k N^{11}_{part/pp*} + l N^{XOR}_{part/pp*}] \tag{56}$$

The first contribution consists of k interactions detected in both modules, l of the remaining $n-k$ interactions detected in only one module and the remaining $n-k-l$ interactions undetected.

The probability of detecting k interactions in both modules is p_{11}^{k}. The probability of detecting l interactions in only one module is p_{XOR}^{l}. The probability of not detecting $n-k-l$ interactions is $(1 - p_{XOR} - p_{11})^{n-k-l}$.

Binomial factors are used to account for all permutations of k out of n interactions and l out of $n-k$ interactions.

The average number of particles given by k interactions detected in both modules is $k N^{11}_{part/pp*}$, while that of l interactions detected in one module is $l N^{XOR}_{part/pp*}$.

Term II Suppose n interactions occurred in an event, Term II can be written as:

$$II = \sum_{k=1}^{n} p_{01}^{k} \binom{n}{k} \left[\sum_{l=1}^{n-k} p_{10}^{l} p_{00}^{n-k-l} \binom{n-k}{l} \right] [kN_{part/pp*}^{01} + lN_{part/pp*}^{10}] \tag{57}$$

The second contribution consists of k interactions detected in module A but not in C, l of the remaining $n-k$ interactions detected in module C but not in A, and the remaining $n-k-l$ interactions undetected.

The probability of detecting k interactions in module A is p_{01}^{k}. The probability of detecting l interactions in module C is p_{10}^{l}. The probability of not detecting $n-k-l$ interactions is p_{00}^{n-k-l}.

Binomial factors are used to account for all permutations of k out of n interactions and l out of $n-k$ interactions.

The average number of particles given by k interactions detected in both modules is $kN_{part/pp*}^{01}$, while that of l interactions detected in one module is $lN_{part/pp*}^{10}$.

Sum over l The l-sums in Equations 56 and 57 can be evaluated by means of the binomial theorem:

$$kN_{part/pp*}^{11} \sum_{l=0}^{n-k} p_{XOR}^{l} \left(1 - p_{XOR} - p_{11}\right)^{n-k-l} \binom{n-k}{l} = kN_{part/pp*}^{11} (1 - p_{11})^{n-k} \tag{58}$$

$$N_{part/pp*}^{XOR} \sum_{l=0}^{n-k} l\ p_{XOR}^{l} \left(1 - p_{XOR} - p_{11}\right)^{n-k-l} \binom{n-k}{l} = N_{part/pp*}^{XOR} (n-k) p_{XOR} (1 - p_{11})^{n-k-1} \tag{59}$$

$$kN_{part/pp*}^{01} \sum_{l=1}^{n-k} p_{10}^{l} p_{00}^{n-k-l} \binom{n-k}{l} = kN_{part/pp*}^{01} \left[(p_{00} + p_{10})^{n-k} - p_{00}^{n-k} \right] \tag{60}$$

$$N_{part/pp*}^{10} \sum_{l=1}^{n-k} l p_{10}^{l}\ p_{00}^{n-k-l} \binom{n-k}{l} = N_{part/pp*}^{10} (n-k)\ p_{10} (p_{00} + p_{10})^{n-k-1} \tag{61}$$

Sum over k Equations 58-61 are used to evaluate the k-sums in Equations 56 and 57 by means of the binomial theorem:

$$N_{part/pp*}^{11} \sum_{k=1}^{n} k p_{11}^{k} (1 - p_{11})^{n-k} \binom{n}{k} = N_{part/pp*}^{11} p_{11} n \tag{62}$$

$$N_{part/pp*}^{XOR} p_{XOR} \sum_{k=1}^{n} n p_{11}^{k} (1 - p_{11})^{n-k-1} \binom{n}{k} = N_{part/pp*}^{XOR} p_{XOR} n \left[\left(\frac{1}{1 - p_{11}} \right) - (1 - p_{11})^{n-1} \right] \tag{63}$$

$$-N_{part/pp*}^{XOR} p_{XOR} \sum_{k=1}^{n} k p_{11}^{k} (1 - p_{11})^{n-k-1} \binom{n}{k} = -N_{part/pp*}^{XOR} p_{XOR} n \frac{p_{11}}{1 - p_{11}} \tag{64}$$

$$N_{part/pp*}^{01} \sum_{k=1}^{n} k p_{01}^{k} (p_{00} + p_{10})^{n-k} \binom{n}{k} = N_{part/pp*}^{01} p_{01} n (p_{00} + p_{01} + p_{10})^{n-1} \tag{65}$$

$$-N_{part/pp*}^{01} \sum_{k=1}^{n} k p_{01}^{k} p_{00}^{n-k} \binom{n}{k} = -N_{part/pp*}^{01} p_{01} n (p_{00} + p_{01})^{n-1} \tag{66}$$

$$N^{10}_{part/pp*}p_{10}\sum_{k=1}^{n} np_{01}^{k}(p_{00}+p_{10})^{n-k-1}\binom{n}{k} = N^{10}_{part/pp*}p_{10}n\left[\frac{(1-p_{11})^{n}}{p_{00}+p_{10}} - (p_{00}+p_{10})^{n-1}\right] \quad (67)$$

$$-N^{10}_{part/pp*}p_{10}\sum_{k=1}^{n} kp_{01}^{k}(p_{00}+p_{10})^{n-k-1}\binom{n}{k} = -N^{10}_{part/pp*}p_{10}np_{01}\frac{(1-p_{11})^{n-1}}{p_{00}+p_{10}} \quad (68)$$

Sum of Terms I and II Using Equation 53, the sum of Equations 62-68 gives:

$$I+II = N^{11}_{part/pp*}p_{11}n + N^{01}_{part/pp*}p_{01}n\left[1-(p_{00}+p_{01})^{n-1}\right] + N^{10}_{part/pp*}p_{10}n\left[1-(p_{00}+p_{10})^{n-1}\right] \quad (69)$$

Poissonian sum The average number of particles per event in coincidence mode is given by the convolution of Equation 69 with a Poissonian of average μ:

$$N^{AND}_{part/BC} = \sum_{n=0}^{\infty}(I+II)\frac{e^{-\mu}\mu^{n}}{n!} \quad (70)$$

Given the relations:

$$\sum_{n=0}^{\infty} n\frac{e^{-\mu}\mu^{n}}{n!} = \mu \qquad \text{and} \qquad \sum_{n=0}^{\infty}\frac{k^{n}}{n!} = e^{k} \quad (71)$$

Equation 70 becomes:

$$N^{AND}_{part/BC} = N^{11}_{part/pp*}p_{11}\mu + N^{01}_{part/pp*}p_{01}\mu\left[1-e^{-\mu(p_{10}+p_{11})}\right] + N^{10}_{part/pp*}p_{10}\mu\left[1-e^{-\mu(p_{01}+p_{11})}\right] \quad (72)$$

Using the following relations:

$$\begin{aligned} N^{01}_{part/pp*}p_{01} &= N^{A}_{part/pp^{*}}\varepsilon_{A} - N^{AND}_{part/pp^{*}}\varepsilon_{AND} = N^{A}_{part/pp} - N^{AND}_{part/pp} \\ N^{10}_{part/pp*}p_{10} &= N^{C}_{part/pp^{*}}\varepsilon_{C} - N^{AND}_{part/pp^{*}}\varepsilon_{AND} = N^{C}_{part/pp} - N^{AND}_{part/pp} \end{aligned} \quad (73)$$

Equation 72 can be written as:

$$\begin{aligned} N^{AND}_{part/BC} = \ & \mu N^{AND}_{part/pp} + \mu\left(N^{A}_{part/pp} - N^{AND}_{part/pp}\right)(1-e^{-\mu\varepsilon_{C}}) \\ & +\mu\left(N^{C}_{part/pp} - N^{AND}_{part/pp}\right)(1-e^{-\mu\varepsilon_{A}}) \end{aligned} \quad (74)$$

The number of particles per event can be extracted from the number of hits per event as it was done for the single side mode:

$$N_{part/BC} = -N_{tubes}\ln\left(1-\frac{N_{hits/BC}}{N_{tubes}}\right) \quad (75)$$

Once again the calculation has to be done for each BC and then averaged.